Skew Bracoids in the Holomorph

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Skew Bracoids

Definition

A *skew (left) bracoid* is a 5-tuple $(G, \cdot, N, \star, \odot)$, where $(G, \cdot)$ and $(N, \star)$ are groups and $\odot$ is a transitive action of G on N for which

$$g \odot (\eta \star \mu) = (g \odot \eta) \star (g \odot e_N)^{-1} \star (g \odot \mu),$$

for all $g \in G$ and $\eta, \mu \in N$.

- We will assume everything is finite.
- We will frequently write $(G, N, \odot)$, or even (G, N), for $(G, \cdot, N, \star, \odot)$.
- We will refer to $(N, \star)$ as the additive group and $(G, \cdot)$ as the multiplicative or acting group.
- We will use S for $\mathrm{Stab}_G(e_N)$.

The γ-function

Definition/Proposition

Let $(G, N, \odot)$ be a skew bracoid and $g \in G$. The map $\gamma : G \to \mathrm{Perm}(N)$ given by

$$^{\gamma(g)}\eta = (g \odot e_N)^{-1}(g \odot \eta)$$

is in fact a group homomorphism with image contained in $\mathrm{Aut}(N)$.

We call this map the γ-*function* of the skew bracoid.

Definition

Let $(G, N, \odot)$ be a skew bracoid. A subgroup M of N is a left ideal of (G, N) if and only if $^{\gamma(g)}\mu \in M$ for all $g \in G$ and all $\mu \in M$. When M is also normal in N, M is an ideal of (G, N).

Skew Bracoids in the Holomorph

Proposition

Let N be a group. We have a correspondence between

- skew bracoids $(G, N, \odot)$,
- and transitive subgroups A of $\mathrm{Hol}(N)$.

Sketch Proof.

Let $(G, N, \odot)$ be a skew bracoid. The image of the map $\Gamma : G \to \mathrm{Hol}(N)$ given by $g \mapsto (g \odot e_N, \gamma(g))$ is a transitive subgroup of $\mathrm{Hol}(N)$.

Conversely any transitive subgroup A of $\mathrm{Hol}(N)$ can be packaged up with N itself to form a skew bracoid $(A, N, \odot)$, with all the obvious operations. □

Strength

Recall that passing to the holomorphs kills off any kernel in the action, leaving the action faithful and the skew bracoid *reduced*.

Proposition

Let $\mathcal{B} = (G, N, \odot)$ and $\mathcal{B}' = (G', N, \odot')$ be reduced skew bracoids, corresponding to A and A' respectively in $\mathrm{Hol}(N)$. We have that $\mathcal{B}$ and $\mathcal{B}'$ are isomorphic as skew bracoids if and only if A is conjugate to A' via an automorphism of N.

Some Familiar Concepts Re-framed

Let (A, N) be a skew bracoid with $A \subseteq \mathrm{Hol}(N)$.

- The stabiliser $\mathrm{Stab}_A(e_N)$ is precisely the purely automorphism elements of A, we denote this S_A.
- Let $[\eta, \alpha] \in A$ and $\mu \in N$. We have

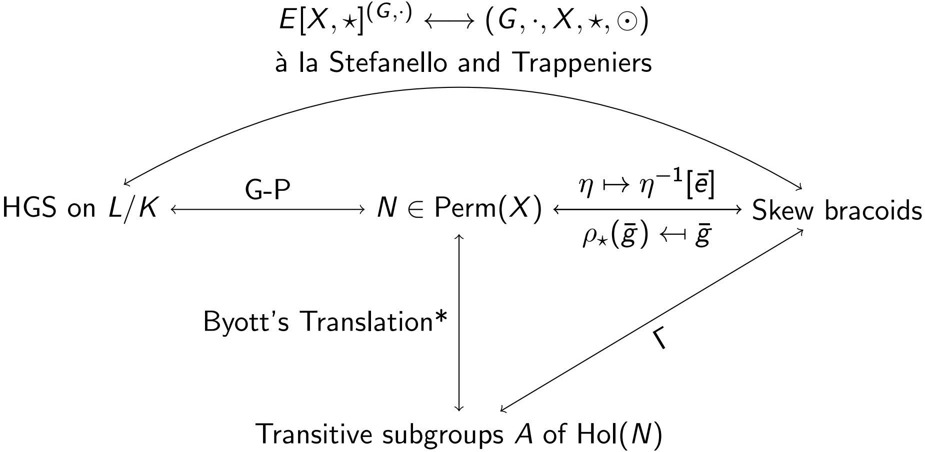
$$\begin{aligned} {}^{\gamma([\eta,\alpha])}\mu &= ([\eta, \alpha]e_N)^{-1}([\eta, \alpha]\mu) \\ &= \eta^{-1}\eta\alpha(\mu) \\ &= \alpha(\mu), \end{aligned}$$

  so the image of γ consists of the automorphisms of N that appear in an element of A.
- A left ideal is then a subgroup M of N such that the automorphisms $\alpha \in \gamma(A) \subseteq \mathrm{Aut}(N)$ restrict to automorphisms of M.

The Various Correspondences

Let L/K be a finite, separable extension of fields with Galois closure E.

Let $G = \mathrm{Gal}(E/K)$ and $S = \mathrm{Gal}(E/L)$, and $X = G/S$.



The Hopf-Galois Correspondence

Again, let L/K be a finite, separable extension of fields with Galois closure E. Let $G = \mathrm{Gal}(E/K)$ and $S = \mathrm{Gal}(E/L)$, and $X = G/S$.

Theorem

Suppose $E[X, \star]^G$ is a Hopf-Galois structure on L/K, corresponding to the skew bracoid $(G, X, \odot)$. Let G' be a subgroup of G, containing S, the following are equivalent

- $E^{G'}$ appears in the image of the HGC for $E[X, \star]^G$,
- $G' \odot e_N$ is a left ideal of $(G, X, \odot)$.

It is already well known that if the structure is almost classical then the HGC is surjective, but other results in the separable case are scarce.

And in the Holomorph?

Translating "$G' \odot e_N$ is a left ideal of $(G, X, \odot)$" to our holomorph formulation with $A = \Gamma(G)$ and $A' = \Gamma(G')$, we would be asking whether

- $A' e_N$ is a subgroup of N,
- for which the elements of $\gamma(A) \subseteq \mathrm{Aut}(N)$ restrict to automorphisms of $A' e_N$.

Qualitative Results

Hopf-Galois Structure	**Skew Bracoid**	**Holomorph**
Almost classical extension	$G \cong H \rtimes S$	$A \cong B \rtimes S_A$
Almost classical structure	Almost classical	$[N, id] \subseteq A$
$E^{G'}$ in image of HGC	$G' \odot e_N$ left ideal	$A' e_N \leq N$, $\gamma(A) \subseteq \mathrm{Aut}(A' e_N)$

An Example

Let $N = \langle \sigma, \tau \ : \ \sigma^p = \tau^q = e, \tau\sigma = \sigma\tau \rangle \cong C_{pq}$ with p, q odd primes and $p = 2q + 1$. Let $\alpha, \beta, \gamma, \delta$ be generators for $\mathrm{Aut}(N)$ where α, β fix τ; δ, γ fix σ; and $\mathrm{ord}(\alpha) = q, \mathrm{ord}(\beta) = 2, \mathrm{ord}(\gamma) = 2^r, \mathrm{ord}(\delta) = s$ $(q - 1 = 2^r s)$.

	Order	Parameters	Group
(1)	$2^{c+1}dpq^2$	$0 \leq c \leq r, d \mid s$	$N \rtimes \langle \alpha, \beta, \gamma^{2^{r-c}}, \delta^{s/d} \rangle$
(2)	$2^c dpq^2$	$0 \leq c \leq r, d \mid s$	$N \rtimes \langle \alpha, \gamma^{2^{r-c}}, \delta^{s/d} \rangle$
(3)	$2^c dpq^2$	$1 \leq c \leq r, d \mid s$	$N \rtimes \langle \alpha, \beta\gamma^{2^{r-c}}, \delta^{s/d} \rangle$
(4)	$2^{c+1}dpq$	$0 \leq c \leq r, d \mid s$	$N \rtimes \langle \beta, \gamma^{2^{r-c}}, \delta^{s/d} \rangle$
(5)	$2^c dpq$	$0 \leq c \leq r, d \mid s$	$N \rtimes \langle \gamma^{2^{r-c}}, \delta^{s/d} \rangle$
(6)	$2^c dpq$	$1 \leq c \leq r, d \mid s$	$N \rtimes \langle \beta\gamma^{2^{r-c}}, \delta^{s/d} \rangle$
(7)	$2pq$	$1 \leq t \leq q - 1$	$\langle \sigma, [\tau, \alpha^t] \rangle \rtimes \langle \beta \rangle$
(8)	pq	$1 \leq t \leq q - 1$	$\langle \sigma, [\tau, \alpha^t] \rangle$

The Tricky Ones

	Order	Parameters	Group
(8)	pq	$1 \leq t \leq q-1$	$\langle \sigma, [\tau, \alpha^t] \rangle$

Let A be as in (8). The proper subgroups of A are then $\{[e, id]\}$, $\langle \sigma \rangle$, $\langle [\tau, \alpha^t] \rangle$ and $\langle [\sigma^i \tau, \alpha^t] \rangle$ for $1 \leq i < p$.

Taking $A' = \langle [\sigma\tau, \alpha^t] \rangle$ and writing σ^g for $\alpha(\sigma)$, we get

$$A' e_N = \left\{ \sigma^{\sum_{j=0}^{k-1} g^j} \tau^k \;\middle|\; 1 \leq k \leq q \right\}$$

which is not a subgroup of N. Hence the Hopf-Galois correspondence is not surjective.

The Tricky Ones

	Order	Parameters	Group
(7)	$2pq$	$1 \leq t \leq q-1$	$\langle \sigma, [\tau, \alpha^t] \rangle \rtimes \langle \beta \rangle$

With A as in (7) however, we are forced to include $\langle \beta \rangle = S_A$ in our A'. This β allows us to disentangle a power of σ from $[\sigma^i \tau, \alpha^t]$ so that we would end up with the whole of A. The proper subgroups of A containing $\langle \beta \rangle$ are then $\langle \beta \rangle$, $\langle \sigma \rangle \rtimes \langle \beta \rangle$, $\langle [\tau, \alpha^t] \rangle \rtimes \langle \beta \rangle$.

In (7) then, the N parts of each possible A' form subgroups of N, and any automorphisms will naturally restrict to these subgroups. Hence the Hopf-Galois correspondence is surjective.

Thank you for your attention!

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